

THE PRISM TABLEAU MODEL FOR SCHUBERT POLYNOMIALS

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ABSTRACT. The Schubert polynomials lift the Schur basis of symmetric polynomials into a basis for $\mathbb{Z}[x_1, x_2, \dots]$. We suggest the *prism tableau model* for these polynomials. A novel aspect of this alternative to earlier results is that it directly invokes semistandard tableaux; it does so as part of a colored tableau amalgam. In the Grassmannian case, a prism tableau with colors ignored is a semistandard Young tableau. Our arguments are developed from the Gröbner geometry of matrix Schubert varieties.

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1. INTRODUCTION

1.1. Overview. A. Lascoux–M.-P. Schützenberger [LaSh82a] recursively defined an integral basis of $\text{Pol} = \mathbb{Z}[x_1, x_2, \dots]$ given by the **Schubert polynomials** $\{\mathfrak{S}_w : w \in S_\infty\}$. If w_0 is the longest length permutation in the symmetric group S_n then $\mathfrak{S}_{w_0} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. Otherwise, $w \neq w_0$ and there exists i such that $w(i) < w(i+1)$. Now one sets $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$, where $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$ (since the polynomial operators ∂_i form a representation of S_n , this definition is self-consistent.) It is true that under the standard inclusion $\iota : S_n \hookrightarrow S_{n+1}$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. Thus one can refer to \mathfrak{S}_w for each $w \in S_\infty = \bigcup_{n \geq 1} S_n$.

Textbook understanding of the ring Sym of symmetric polynomials centers around the basis of Schur polynomials and its successful companion, the theory of Young tableaux. Since Schur polynomials are instances of Schubert polynomials, the latter basis naturally lifts the Schur basis into Pol . Yet, it is also true that Schubert polynomials have nonnegative integer coefficients. Consequently, one has a natural problem:

Is there a combinatorial model for Schubert polynomials that is analogous to the semistandard tableau model for Schur polynomials?

Indeed, multiple solutions have been discovered over the years, e.g., [Ko90], [BiJoSt93], [BeBi93], [FoSt94], [FoKi96], [FoGrReSh97], [Ma98], [BeSo98, BeSo02], [BuKrTaYo04] and [CoTa13] (see also [LaSh85]). In turn, the solutions [BiJoSt93, BeBi93, FoSt94, FoKi96] have been the foundation for a vast literature at the confluence of combinatorics, representation theory and combinatorial algebraic geometry.

We wish to put forward another solution – a novel aspect of which is that it directly invokes semistandard tableaux. Both the statement and proof of our alternative model build upon ideas about the Gröbner geometry of matrix Schubert varieties X_w . We use the Gröbner degeneration of X_w and the interpretation of \mathfrak{S}_w as mutidegrees of X_w [KnMi05]. Actually, a major purpose of *loc. cit.* is to establish the geometric naturality of the combinatorics of [BiJoSt93, BeBi93, FoKi96]. Our point of departure is stimulated by later work of A. Knutson on *Frobenius splitting* [Kn09, Theorem 6 and Section 7.2].

1.2. The main result. We recall some permutation combinatorics found in, e.g., in [Ma01]. The **diagram** of w is $D(w) = \{(i, j) : 1 \leq i, j \leq n, w(i) > j \text{ and } w^{-1}(j) > i\} \subset n \times n$. Let $\mathcal{E}ss(w) \subset D(w)$ be the **essential set** of w : the southeast-most boxes of each connected component of w . The **rank function** is $r_w(i, j) = \#\{t \leq i : w(t) \leq j\}$.

Define w to be **Grassmannian** if it has at most one descent, i.e., at most one index k such that $w(k) > w(k+1)$. If in addition w^{-1} is Grassmannian then w is **biGrassmannian**. For $e = (i, j) \in \mathcal{E}ss(w)$ let R_e be the $(i - r_w(i, j)) \times (j - r_w(i, j))$ rectangle with southwest corner at position $(i, 1)$ of $n \times n$. The **shape** of w is $\lambda(w) = \bigcup_{e \in \mathcal{E}ss(w)} R_e$:

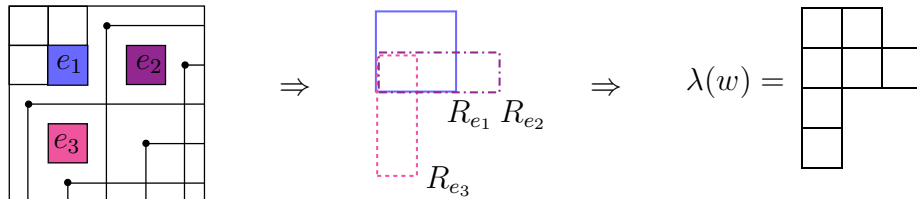


FIGURE 1. The diagram of $w = 35142$ (with color coded essential set $\{e_1, e_2, e_3\}$), the overlay of $R_{e_1}, R_{e_2}, R_{e_3}$, and the shape $\lambda(w)$.

A **prism tableau** T for w fills $\lambda(w)$ with colored labels (one color for each $e \in \mathcal{E}_{ss}(w)$) such that the labels of color e :

- (S1) sit in a box of R_e ;
- (S2) weakly decrease along rows from left to right;
- (S3) strictly increase along columns from top to bottom; and
- (S4) are **flagged**: a label is no bigger than the row of the box it sits in.

Let $d_i(w)$ be the number of distinct values (ignoring color) seen on the i -th antidiagonal (i.e., the one meeting $(i, 1)$), for $i = 1, 2, \dots, n$. We say T is **minimal** if $\sum_{i=1}^n d_i(w) = \ell(w)$, where $\ell(w)$ is the Coxeter length of w .

Let ℓ_c be a label ℓ of color c . Labels $\{\ell_c, \ell_d, \ell'_e\}$ in the same antidiagonal form an **unstable triple** if $\ell < \ell'$ and replacing the ℓ_c with ℓ'_e gives a prism tableau. See Example 1.3. Let $\text{Prism}(w)$ be the set of minimal prism tableaux with no unstable triples. Finally, set

$$\mathfrak{P}_w(x_1, \dots, x_n) := \sum_{T \in \text{Prism}(w)} \text{wt}(T), \text{ where } \text{wt}(T) = \prod_i x_i^{\# \text{ of antidiagonals containing } i}.$$

Theorem 1.1. $\mathfrak{S}_w(x_1, \dots, x_n) = \mathfrak{P}_w(x_1, \dots, x_n)$.

Example 1.2 (Reduction to semistandard tableaux). Consider the Grassmannian permutation $w = 246135$. Conflating prism tableaux with their weights, Theorem 1.1 asserts:

$$\mathfrak{S}_w = \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 2 \\ \hline 333 & 33 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 2 \\ \hline 333 & 33 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 2 \\ \hline 333 & 33 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 1 \\ \hline 333 & 22 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 1 \\ \hline 333 & 22 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 1 \\ \hline 333 & 33 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 1 \\ \hline 333 & 33 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 22 & 1 \\ \hline 333 & 33 & 3 \\ \hline \end{array}.$$

Forgetting colors gives the following expansion of the Schur polynomial:

$$s_{\lambda(w)} = \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 2 \\ \hline 3 & 3 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 2 \\ \hline 3 & 3 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 1 \\ \hline 3 & 2 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 1 \\ \hline 3 & 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 1 \\ \hline 3 & 3 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 1 \\ \hline 3 & 3 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 1 \\ \hline 2 & 1 \\ \hline 3 & 3 & 3 \\ \hline \end{array}.$$

In general, if w is Grassmannian then $\lambda(w)$ is a (French) Young diagram. Moreover, each cell of $T \in \text{Prism}(w)$ uses only one number. (See Lemma 4.1.) Replacing each set in T by the common value gives a *reverse* semistandard tableau. Thus $\mathfrak{P}_w = s_{\lambda(w)}$ follows. \square

Prism tableaux provide a means to understand the RC -graphs of [BeBi93, FoKi96]. We think of the $\#\mathcal{E}_{ss}(w)$ -many semistandard tableaux of a prism tableau T as the “dispersion” of the associated RC -graph through T . See Sections 4.1 and 4.3.

Minimality and the unstable triple condition bond the tableau of each color, which is one reason why we prefer not to think of a prism tableau as merely a $\#\mathcal{E}_{ss}(w)$ -tuple:

Example 1.3 (Unstable triples). Let $w = 42513$. Then $\#\mathcal{E}_{ss}(w) = 3$. The minimal prism tableaux and their weights are:

T	<table><tr><td>11</td><td>1</td><td>1</td></tr><tr><td>22</td><td>1</td><td></td></tr><tr><td>33</td><td>3</td><td></td></tr></table>	11	1	1	22	1		33	3		<table><tr><td>11</td><td>1</td><td>1</td></tr><tr><td>21</td><td>1</td><td></td></tr><tr><td>33</td><td>3</td><td></td></tr></table>	11	1	1	21	1		33	3		<table><tr><td>11</td><td>1</td><td>1</td></tr><tr><td>22</td><td>1</td><td></td></tr><tr><td>33</td><td>2</td><td></td></tr></table>	11	1	1	22	1		33	2		<table><tr><td>11</td><td>1</td><td>1</td></tr><tr><td>21</td><td>1</td><td></td></tr><tr><td>33</td><td>2</td><td></td></tr></table>	11	1	1	21	1		33	2	
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$\text{wt}(T)$	$x_1^3 x_2 x_3^2$	$x_1^3 x_2 x_3^2$	$x_1^3 x_2^2 x_3$	$x_1^3 x_2^2 x_3$																																				

The second and the fourth tableaux have an unstably paired label. In both tableaux, the pink 1 in the second antidiagonal is replaceable by a pink 2. So $\mathfrak{S}_w = x_1^3 x_2 x_3^2 + x_1^3 x_2^2 x_3$. \square

\mathfrak{S}_{1234}	\emptyset	1	\mathfrak{S}_{3124}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline \end{array}$	x_1^2
\mathfrak{S}_{1243}	$\begin{array}{ c } \hline 1 \\ \hline \end{array} + \begin{array}{ c } \hline 2 \\ \hline \end{array} + \begin{array}{ c } \hline 3 \\ \hline \end{array}$	$x_1 + x_2 + x_3$	\mathfrak{S}_{3142}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 1 & \\ \hline 2 & \end{array} + \begin{array}{ c c } \hline 1 & 1 \\ \hline 1 & \\ \hline 3 & \end{array}$	$x_1^2 x_2 + x_1^2 x_3$
\mathfrak{S}_{1324}	$\begin{array}{ c } \hline 1 \\ \hline \end{array} + \begin{array}{ c } \hline 2 \\ \hline \end{array}$	$x_1 + x_2$	\mathfrak{S}_{3214}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & \end{array}$	$x_1^2 x_2$
\mathfrak{S}_{1342}	$\begin{array}{ c } \hline 2 \\ \hline 3 \end{array} + \begin{array}{ c } \hline 1 \\ \hline 2 \end{array} + \begin{array}{ c } \hline 1 \\ \hline 3 \end{array}$	$x_2 x_3 + x_1 x_2 + x_1 x_3$	\mathfrak{S}_{3241}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \end{array}$	$x_1^2 x_2 x_3$
\mathfrak{S}_{1423}	$\begin{array}{ c c } \hline 2 & 2 \\ \hline \end{array} + \begin{array}{ c c } \hline 1 & 1 \\ \hline \end{array} + \begin{array}{ c c } \hline 2 & 1 \\ \hline \end{array}$	$x_2^2 + x_1^2 + x_1 x_2$	\mathfrak{S}_{3412}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & 2 \end{array}$	$x_1^2 x_2^2$
\mathfrak{S}_{1432}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & \end{array} + \begin{array}{ c c } \hline 1 & 1 \\ \hline 3 & \end{array} + \begin{array}{ c c } \hline 2 & 1 \\ \hline 2 & \end{array} + \begin{array}{ c c } \hline 2 & 1 \\ \hline 3 & \end{array} + \begin{array}{ c c } \hline 2 & 2 \\ \hline 3 & \end{array}$	$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3$	\mathfrak{S}_{3421}	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \end{array}$	$x_1^2 x_2^2 x_3$
\mathfrak{S}_{2134}	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	x_1	\mathfrak{S}_{4123}	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline \end{array}$	x_1^3
\mathfrak{S}_{2143}	$\begin{array}{ c } \hline 1 \\ \hline 1 \end{array} + \begin{array}{ c } \hline 1 \\ \hline 2 \end{array} + \begin{array}{ c } \hline 1 \\ \hline 3 \end{array}$	$x_1^2 + x_1 x_2 + x_1 x_3$	\mathfrak{S}_{4132}	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 1 & & \\ \hline 2 & & \end{array} + \begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 1 & & \\ \hline 3 & & \end{array}$	$x_1^3 x_2 + x_1^3 x_3$
\mathfrak{S}_{2314}	$\begin{array}{ c } \hline 1 \\ \hline 2 \end{array}$	$x_1 x_2$	\mathfrak{S}_{4213}	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 2 & & \end{array}$	$x_1^3 x_2$
\mathfrak{S}_{2341}	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline 3 \end{array}$	$x_1 x_2 x_3$	\mathfrak{S}_{4231}	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 2 & & \\ \hline 3 & & \end{array}$	$x_1^3 x_2 x_3$
\mathfrak{S}_{2413}	$\begin{array}{ c } \hline 1 \\ \hline 2 & 2 \end{array} + \begin{array}{ c } \hline 1 \\ \hline 2 & 2 \end{array}$	$x_1 x_2^2 + x_1^2 x_2$	\mathfrak{S}_{4312}	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 2 & 2 & \end{array}$	$x_1^3 x_2^2$
\mathfrak{S}_{2431}	$\begin{array}{ c } \hline 1 \\ \hline 2 & 2 \\ \hline 3 \end{array} + \begin{array}{ c } \hline 1 \\ \hline 2 & 2 \end{array}$	$x_1 x_2^2 x_3 + x_1^2 x_2 x_3$	\mathfrak{S}_{4321}	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \end{array}$	$x_1^3 x_2^2 x_3$

TABLE 1. $\text{Prism}(w)$ and \mathfrak{S}_w for all $w \in S_4$

1.3. **Organization.** In Section 2 we present the general geometric perspective behind the rule and its proof. In the case at hand, we need to study the Stanley-Reisner simplicial complex associated to the Gröbner limit of X_w ; this is done in Section 3. In Section 4, we collect some additional results and remarks.

2. MAIN IDEA OF THE MODEL AND ITS PROOF

Let $G = \mathrm{GL}_n$ and B and B^+ the Borel subgroups of lower and upper triangular matrices in G . Identify the **flag variety** with the coset space $B \backslash G$. Let T be the maximal torus in B . Suppose $\mathfrak{X} \subset B \backslash G$ is an arbitrary subvariety and $\pi : G \twoheadrightarrow B \backslash G$ is the natural projection. Then

$$X = \overline{\pi^{-1}(\mathfrak{X})} \subseteq \mathrm{Mat}_{n \times n}$$

carries a left B action and thus the action of T . Therefore, one can speak of the equivariant cohomology class

$$[X]_T \in H_T(\mathrm{Mat}_{n \times n}) \cong \mathbb{Z}[x_1, \dots, x_n].$$

Moreover, the polynomial $[X]_T$ is a coset representative under the Borel presentation of

$$[\mathfrak{X}] \in H^*(B \backslash G, \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n] / I^{S_n},$$

where I^{S_n} is the ideal generated by (non-constant) elementary symmetric polynomials. This is a key perspective of work of A. Knutson-E. Miller [KnMi05] when \mathfrak{X} is a Schubert variety.

Let $Y \subseteq \mathrm{Mat}_{n \times n}$ be an equidimensional, reduced union of coordinate subspaces. Given $\mathcal{P} \subset n \times n$, we represent \mathcal{P} visually as a collection of $+$'s in the $n \times n$ grid. We say \mathcal{P} is a **plus diagram** for Y , if

$$\mathcal{L}_{\mathcal{P}} := \{M \in \mathrm{Mat}_{n \times n} : M_{i,j} = 0 \text{ if } (i,j) \in \mathcal{P}\} \subset Y.$$

Let $\mathrm{Plus}(Y)$ be the set of all such plus diagrams. Let $\mathrm{MinPlus}(Y)$ be the set of minimal plus diagrams, i.e., those \mathcal{P} for which removing any $+$ would not return an element of $\mathrm{Plus}(Y)$. We refer to the union of plus diagrams as an **overlay** to emphasize whenever (i,j) is in \mathcal{P} or \mathcal{P}' , the diagram for $\mathcal{P} \cup \mathcal{P}'$ also has a $+$ in position (i,j) .

Each \mathcal{P} corresponds 1 : 1 to a face of the Stanley-Reisner complex Δ_Y . Let $\Delta_{n \times n}$ be the power set of $\{(i,j) : 1 \leq i, j \leq n\}$. Then $\Delta_Y \subseteq \Delta_{n \times n}$ and for each \mathcal{P} one has the face

$$\mathcal{F}_{\mathcal{P}} = \{(i,j) : 1 \leq i, j \leq n \text{ and } (i,j) \notin \mathcal{P}\}.$$

The faces of Δ_Y are ordered by reverse containment of their plus diagrams. Thus, facets (maximal dimensional faces) of Δ_Y coincide with elements of $\mathrm{MinPlus}(Y)$. In addition, taking the overlay of $\mathcal{P} \in \mathrm{Plus}(Y)$ and $\mathcal{Q} \in \mathrm{Plus}(Z)$ corresponds to intersecting faces in the Stanley-Reisner complex:

$$\mathcal{F}_{\mathcal{P} \cup \mathcal{Q}} = \mathcal{F}_{\mathcal{P}} \cap \mathcal{F}_{\mathcal{Q}} \in \Delta_Y \cap \Delta_Z.$$

Through the interpretation of $[Y]_T$ as a *multidegree*, one may express $[Y]_T$ as a generating series over $\mathrm{MinPlus}(Y)$. That is,

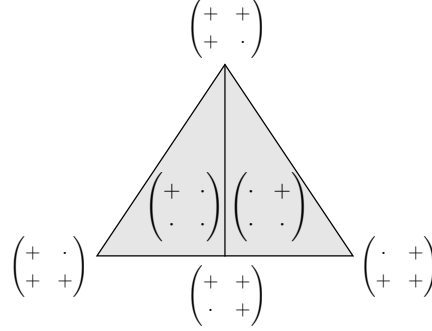
$$(2.1) \quad [Y]_T = \sum_{\mathcal{P} \in \mathrm{MinPlus}(Y)} \mathrm{wt}(\mathcal{P}), \text{ where } \mathrm{wt}(\mathcal{P}) = \prod_{i=1}^n x_i^{\# \text{ of } + \text{'s in row } i}.$$

For details, the reader may consult [MiSt05]; see Chapter 1 and Chapter 8 (and its notes).

Example 2.1. Let $Y \subset \text{Mat}_{2 \times 2}$ be the zero locus of $z_{1,1}z_{1,2}$, i.e., the union of two coordinate hyperplanes $\{z_{1,1} = 0\} \cup \{z_{1,2} = 0\}$. Then

$$\begin{pmatrix} + & \cdot \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & + \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} + & + \\ \cdot & \cdot \end{pmatrix} \in \text{Plus}(Y)$$

(the first two are in $\text{MinPlus}(Y)$). The complex Δ_Y is the 2-dimensional ball depicted below.



Here $[Y]_T = 2x_1$. □

Suppose \prec is any term order on $\mathbb{C}[\text{Mat}_{n \times n}]$ and $X' := \text{init}_{\prec} X$. Since X is T-stable the same is true of X' ; thus $[X']_T$ is defined. Gröbner degeneration preserves the T-equivariant class, so $[X]_T = [X']_T$. Suppose X' is reduced, and hence a reduced union of coordinate subspaces. Since \mathfrak{X} was assumed to be irreducible, then X is irreducible. So by [KaSt95, Theorem 1] the Stanley-Reisner complex $\Delta_{X'}$ of X' is equidimensional. Hence we may apply the discussion above using $Y = X'$ to compute $[X']_T = [X]_T$.

We are interested in understanding $\Delta_{X'}$ under certain hypotheses on X . Assume that we have a collection of varieties $X, X_1, \dots, X_m \subseteq V \cong \mathbb{C}^N$ such that

$$(2.2) \quad X = X_1 \cap X_2 \cap \dots \cap X_k.$$

Assume \prec is a term order on $\mathbb{C}[V]$ that defines a Gröbner degeneration of these varieties so that each Gröbner limit

$$(2.3) \quad X' := \text{init}_{\prec} X, X'_i := \text{init}_{\prec} X_i \text{ (for } i = 1, 2, \dots, k) \text{ is reduced.}$$

Finally, suppose

$$(2.4) \quad X' = X'_1 \cap X'_2 \cap \dots \cap X'_k.$$

Call $\{X_i\}$ a \prec -**spectrum** for X .

To construct a cheap example, pick any Grobner basis $G = \{g_1, \dots, g_M\}$ with square-free lead terms to define X . Now partition $G = G_1 \cup G_2 \cup \dots \cup G_s$ and set X_k to be cut out by G_k . On the other hand, a motivating example is A. Knutson [Kn09, Theorem 6]: given a term order \prec (satisfying a hypothesis), there is a stratification of V into a poset of varieties (ordered by inclusion) with the additional feature that each stratum X admits a \prec -spectrum using higher strata.

How can a \prec -spectrum be used to understand the combinatorics of $[X']_T$? Here is a simple observation:

Lemma 2.2. *Let $\{X_i\}$ be a \prec -spectrum for X . Then*

- (I) $\text{Plus}(X') = \{\mathcal{P}_1 \cup \dots \cup \mathcal{P}_k : \mathcal{P}_i \in \text{Plus}(X'_i)\}$
- (II) $\text{MinPlus}(X') \subseteq \{\mathcal{P}_1 \cup \dots \cup \mathcal{P}_k : \mathcal{P}_i \in \text{MinPlus}(X'_i)\}$

Proof. (I): Let $\mathcal{P} \in \text{Plus}(X')$. Then $\mathcal{L}_{\mathcal{P}} \subseteq X' \subseteq X'_i$ for all i . Therefore $\mathcal{P} \in \text{Plus}(X'_i)$ and trivially $\mathcal{P} = \mathcal{P} \cup \dots \cup \mathcal{P}$, proving “ \subseteq ”. For the other containment, suppose $\mathcal{P}_i \in \text{Plus}(X'_i)$ for $1 \leq i \leq k$ and let $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$. Then $\mathcal{L}_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}_1} \cap \dots \cap \mathcal{L}_{\mathcal{P}_k}$ and hence $\mathcal{L}_{\mathcal{P}} \subseteq \mathcal{L}_{\mathcal{P}_i} \subseteq X'_i$. So $\mathcal{P} \in \text{Plus}(X'_i)$ for each i , which implies $\mathcal{P} \in \text{Plus}(X')$.

(II): Let $\mathcal{P} \in \text{MinPlus}(X')$. By (I), $\mathcal{P} \in \text{Plus}(X'_i)$ for each i . Then there exists $\mathcal{P}_i \in \text{MinPlus}(X'_i)$ so that $\mathcal{P}_i \subseteq \mathcal{P}$. Then $\mathcal{P} \supseteq \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k \in \text{Plus}(X')$ by (I). As \mathcal{P} is minimal, this is an equality. \square

Our point is that in good cases, the plus diagrams of X'_i are “simpler” to understand than those of X . Lemma 2.2(II) says that one can think of each $\mathcal{P} \in \text{MinPlus}(X)$ as an overlay $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$ of these simpler \mathcal{P}_i . Of course, this representation is not unique in general, so one can make a *choice* of representation for each \mathcal{P} . The hope is to transfer understanding of the combinatorics of $\text{MinPlus}(X_i)$ to the combinatorics of $\text{MinPlus}(X)$.

3. PROOF OF THE THEOREM 1.1

We now carry out the ideas described in Section 2 in the case of Schubert varieties.

3.1. Matrix Schubert varieties and Schubert polynomials. The flag variety $B \backslash G$ decomposes into **Schubert cells** $\mathfrak{X}_w^\circ := B \backslash BwB^+$ indexed by $w \in S_n$. The **Schubert variety** is the Zariski-closure $\mathfrak{X}_w := \overline{\mathfrak{X}_w^\circ}$. The **matrix Schubert variety** is

$$X_w := \overline{\pi^{-1}(\mathfrak{X}_w)} \subset \text{Mat}_{n \times n}.$$

Let $Z = (z_{ij})_{1 \leq i, j \leq n}$ be the generic $n \times n$ matrix. The **Schubert determinantal ideal** is

$$I_w = \langle r_w(i, j) + 1 \text{ minors of the the northwest } i \times j \text{ submatrix of } Z \rangle \subset \mathbb{C}[\text{Mat}_{n \times n}].$$

In [Fu91, Lemma 3.10] it is proved that I_w cuts out X_w scheme-theoretically. Moreover in *loc. cit.* it is shown that I_w is generated by the smaller set of generators coming from those $(i, j) \in \mathcal{E}_{ss}(w)$.

By [KnMi05, Theorem A],

$$[X_w]_{\mathbb{T}} = \mathfrak{S}_w(x_1, \dots, x_n) \in H_{\mathbb{T}}(\text{Mat}_{n \times n}).$$

Moreover, let \prec_{anti} be any **antidiagonal term order** on $\mathbb{C}[\text{Mat}_{n \times n}]$, i.e., one that picks off the antidiagonal term of any minor of Z . In [KnMi05, Theorem B] it is shown that $\text{MinPlus}(X'_w)$ are in a transparent bijection with the *RC*-graphs of [BeBi93] (cf. [FoKi96]).

For each $e \in \mathcal{E}_{ss}(w)$, there is a unique biGrassmannian permutation u_e such that $r_{u_e}(e) = r_w(e)$ and $\mathcal{E}_{ss}(u_e) = \{e\}$ [LaSh96]. Let

$$\text{biGrass}(w) := \{u_e : e \in \mathcal{E}_{ss}(w)\} = \{u_1, \dots, u_k\}.$$

Call $\{X_{u_1}, \dots, X_{u_k}\}$ the **biGrassmannian \prec_{anti} -spectrum** for X_w . By [Kn09, Section 7.2], $\{X_{u_i}\}$ indeed gives a \prec_{anti} -spectrum for X_w over \mathbb{Q} . This result can also be readily obtained (over \mathbb{Z}) if one assumes the Gröbner basis result [KnMi05, Theorem B]. (It should be emphasized that one of the points of [Kn09, Section 7.2] is to reprove said Gröbner basis theorem more easily.)

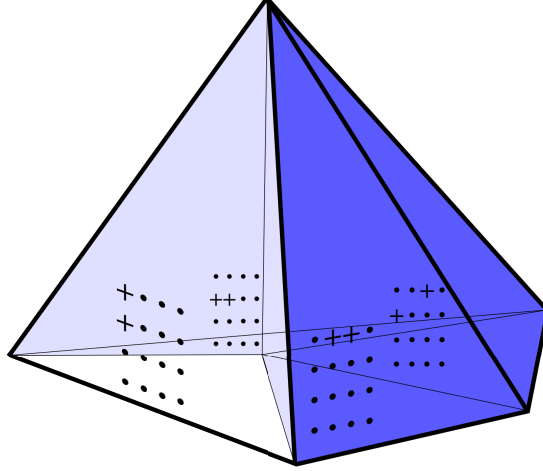


FIGURE 2. The Stanley-Reisner complexes for X'_{1423} and X'_{2314} intersect to give the complex for X'_{2413} . These complexes are a multicone over the depicted complex.

Example 3.1. $X = X_{2413}$ has biGrassmannian \prec_{anti} -spectrum $\{X_1 = X_{1423}, X_2 = X_{2314}\}$. Here $\mathbb{C}[\text{Mat}_{n \times n}] = \mathbb{C}[z_{i,j} : 1 \leq i, j \leq 4]$ and one can check:

$$I_{u_1} = \left\langle \begin{vmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{vmatrix}, \begin{vmatrix} z_{1,1} & z_{1,3} \\ z_{2,1} & z_{2,3} \end{vmatrix}, \begin{vmatrix} z_{1,2} & z_{1,3} \\ z_{2,2} & z_{2,3} \end{vmatrix} \right\rangle, \quad I_{u_2} = \langle z_{1,1}, z_{2,1} \rangle, \quad I_w = I_{u_1} + I_{u_2}.$$

The \prec_{anti} -Gröbner limits are defined by

$$I'_{u_1} = \langle z_{2,1}z_{1,2}, z_{2,1}z_{1,3}, z_{2,2}z_{1,3} \rangle, \quad I'_{u_2} = \langle z_{1,1}, z_{2,1} \rangle, \quad I'_w = I'_{u_1} + I'_{u_2}.$$

Since the prime decomposition of I'_{u_1} is

$$I'_{u_1} = \langle z_{2,1}, z_{2,2} \rangle \cap \langle z_{2,1}, z_{1,3} \rangle \cap \langle z_{1,2}, z_{1,3} \rangle,$$

the facets of $\Delta_{X'_1}$ are labeled by:

$$(3.1) \quad \text{MinPlus}(X'_1) = \left\{ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & + & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & + & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

In Figure 2, these correspond to the indicated tetrahedra.

Similarly, there is a single facet for X'_2 associated to the prime ideal I_{u_2} , labeled by:

$$(3.2) \quad \left\{ \begin{bmatrix} + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

This facet corresponds to the remaining tetrahedron.

There are precisely two minimal overlays of the plus diagrams of (3.1) with the plus diagram of (3.2):

$$\left\{ \begin{bmatrix} + & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} + & \cdot & + & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

This agrees with the prime decomposition $I'_w = \langle z_{1,1}, z_{1,3}, z_{2,1} \rangle \cap \langle z_{1,1}, z_{2,1}, z_{2,2} \rangle$. Geometrically, these label the facets of $\Delta_{X'}$, pictured as light blue triangles in Figure 2.

Finally, applying the discussion of Section 2 (cf. (2.1)) we see that

$$\mathfrak{S}_{u_1} = x_2^2 + x_1x_2 + x_1^2, \quad \mathfrak{S}_{u_2} = x_1x_2, \quad \text{and} \quad \mathfrak{S}_w = x_1x_2^2 + x_1^2x_2$$

(where the terms in each Schubert polynomial correspond respectively to the plus diagrams listed above). \square

Example 3.2 (Digression: diagonal term orders). Fix $w = 2143 \in S_4$ and let \prec_{diag} be any **diagonal term order** on $\mathbb{C}[\text{Mat}_{n \times n}]$, i.e., any order that picks the diagonal term of a minor as the lead term. One has that $X_{2143} = X_{2134} \cap X_{1243}$ (reduced intersection). Now

$$I'_{2134} = \langle z_{11} \rangle \text{ and } I'_{1243} = \langle z_{11}z_{22}z_{33} \rangle.$$

However,

$$I'_{2143} = \langle z_{11}, z_{12}z_{21}z_{33} \rangle \neq I'_{2134} + I'_{1243} = \langle z_{11}, z_{11}z_{22}z_{33} \rangle = I'_{2134}.$$

So $\{X_{2134}, X_{1243}\}$ is not a \prec_{diag} -spectrum for X_w . \square

A permutation is **vexillary** if it is 2143-avoiding; see [Ma01, Section 2.2.1] for details. The following is not needed in the proof of Theorem 1.1:

Proposition 3.3. $\{X_{u_1}, \dots, X_{u_k}\}$ is a \prec_{diag} -spectrum for X_w if and only if w is vexillary.

Proof. Assume w is vexillary. Then by [KnMiYo09, Section 1.4], the essential minors define a \prec_{diag} -Gröbner basis for I_w . The same is true of I_{u_i} since u_i is biGrassmannian and therefore also vexillary. Since the (Gröbner) essential minors of I_w are the concatenation of the (Gröbner) essential minors of the I_{u_i} 's, the spectrum claim follows.

For the converse, assume w is not vexillary, but $\{X_{u_1}, \dots, X_{u_k}\}$ is a \prec_{diag} -spectrum for X_w . Again, we know the essential minors of I_{u_i} form a \prec_{diag} -Gröbner basis. By the spectrum assumption, the concatenation of these k -many Gröbner basis is a \prec_{diag} -Gröbner basis for I_w . However this concatenated Gröbner basis is the set of essential generators for I_w . This directly contradicts [KnMiYo09, Theorem 6.1]. \square

3.2. Multi-plus diagrams. The technical core of our proof is to analyze the combinatorics of overlays of plus diagrams for the biGrassmannian \prec_{anti} -spectrum $\{X_{u_1}, \dots, X_{u_k}\}$. Let

$$\text{Multi}(w) = \prod_{i=1}^k \text{MinPlus}(X'_{u_i})$$

be the set of **multi-plus diagrams** for w : we represent $(\mathcal{P}_1, \dots, \mathcal{P}_k) \in \text{Multi}(w)$ as a placement of colored $+$'s in a single $n \times n$ grid, where (a, b) has a $+$ of color u_i if $(a, b) \in \mathcal{P}_i$.

By Lemma 2.2(I), there is a map

$$\text{supp} : \text{Multi}(w) \rightarrow \text{Plus}(X'_w)$$

given by $(\mathcal{P}_1, \dots, \mathcal{P}_k) \mapsto \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$. Call $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$ the **support** of $(\mathcal{P}_1, \dots, \mathcal{P}_k)$. Central to our study is

$$\text{Multi}(\mathcal{P}) := \text{supp}^{-1}(\mathcal{P}).$$

Example 3.4. Let $w = 42513$. Then $\text{biGrass}(w) = \{41235, 23415, 14523\}$. Now,

$$\mathcal{P} = \begin{bmatrix} + & + & + & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in \text{MinPlus}(X'_w).$$

One can check that

$$\text{Multi}(\mathcal{P}) = \left\{ \begin{bmatrix} ++ & ++ & ++ & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ ++ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} ++ & + & ++ & \cdot & \cdot \\ ++ & \cdot & + & \cdot & \cdot \\ ++ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

□

3.3. Local moves on plus diagrams. A **southwest move** is the following local operation on a plus diagram:

$$(3.3) \quad \begin{bmatrix} \cdot & + \\ \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix}.$$

The inverse operation is a **northeast move**:

$$(3.4) \quad \begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix} \mapsto \begin{bmatrix} \cdot & + \\ \cdot & \cdot \end{bmatrix}.$$

Suppose $\mathcal{E}_{ss}(u) = \{(i, j)\}$. Define $\mathcal{D}_{\text{bot}}(u) \in \text{MinPlus}(X'_u)$ as the $(i - r_u(i, j)) \times (j - r_u(i, j))$ rectangle of $+$'s, with southwest corner in row i and column 1. The following is well-known, and is a consequence (by specialization) of the chute and ladder moves of [BeBi93, Theorem 3.7]:

Lemma 3.5. *Let $u \in S_n$ be biGrassmannian.*

- (I) $\text{MinPlus}(X'_u)$ is connected and closed under the moves (3.3) and (3.4).
- (II) Each $\mathcal{P} \in \text{MinPlus}(X'_u)$ can be obtained from $\mathcal{D}_{\text{bot}}(u)$ using only the moves (3.4).

Define a partial order on $\text{MinPlus}(X'_u)$ by taking the transitive closure of the covering relation $\mathcal{P} < \mathcal{P}'$ if \mathcal{P}' is obtained from \mathcal{P} by a northeast local move (3.4). Let $<'$ be the partial order on $\text{Multi}(w)$ defined as the $\mathcal{E}_{ss}(w)$ -factor Cartesian product of $<$. That is $(\mathcal{P}_1, \dots, \mathcal{P}_k) <' (Q_1, \dots, Q_k)$ if and only if $\mathcal{P}_i < Q_i$ for each i . Then $<'$ induces a partial order on $\text{Multi}(\mathcal{P}) \subseteq \text{Multi}(w)$.

Given $(\mathcal{P}_1, \dots, \mathcal{P}_m) \in \text{Multi}(w)$, a **long move** is a repeated application of (3.3) (respectively, (3.4)) to a single $+$ appearing in one of the \mathcal{P}_i 's. Recall that a **lattice** is a partially ordered set in which every two elements x and y have a least upper bound (join) $x \vee y$.

(join) and a unique greatest lower bound $x \wedge y$ (meet). It is basic that a Cartesian product of lattices is a lattice.

Theorem 3.6. *Let $w \in S_n$ and $\mathcal{P} \in \text{MinPlus}(X'_w)$.*

- (I) *$\text{Multi}(\mathcal{P})$ is connected by long moves.*
- (II) *Each $(\text{MinPlus}(X'_{u_i}), <)$ is a lattice. Consequently, $(\text{Multi}(w), <')$ is a lattice.*
- (III) *$(\text{Multi}(\mathcal{P}), <')$ is a sublattice of $(\text{Multi}(w), <')$.*

Example 3.7. Let $w = 5361724$. Fix

$$\mathcal{P} = \begin{bmatrix} + & + & + & + & \cdot & \cdot & \cdot \\ + & + & \cdot & + & \cdot & \cdot & \cdot \\ + & + & \cdot & + & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in \text{MinPlus}(X'_w).$$

Here $\text{biGrass}(w) = \{\textcolor{violet}{5123467}, \textcolor{red}{3451267}, \textcolor{blue}{1562347}, \textcolor{teal}{1345627}, \textcolor{orange}{1256734}\}$. Figure 3 shows the Hasse diagram for $\text{Multi}(\mathcal{P})$. The poset is a lattice, agreeing with Theorem 3.6(III). \square

We use the following ordering of the $+$'s of $\mathcal{D}_{\text{bot}}(u)$.

$$(3.5) \quad \begin{bmatrix} +_7 & \cdot & \cdot & \\ +_4 & +_8 & \cdot & \cdot \\ +_2 & +_5 & +_9 & \cdot & \cdot \\ +_1 & +_3 & +_6 & +_{10} \end{bmatrix}.$$

In words, order the $+$'s along diagonals, from northwest to southeast, where $+_1$ is at the southwest corner of \mathcal{D}_{bot} .

In view of Lemma 3.5(II), there is a bijection between the $+$'s of $\mathcal{D}_{\text{bot}}(u)$ and any $\mathcal{P} \in \text{MinPlus}(X'_u)$. Hence the ordering (3.5) induces an ordering $+_1, +_2, \dots$ of the $+$'s of \mathcal{P} .

The following two lemmas hold by Lemma 3.5 and an easy induction.

Lemma 3.8. *If in $\mathcal{D}_{\text{bot}}(u) \in \text{MinPlus}(X'_u)$ the label $+_a$ is weakly southwest of $+_b$, then the same is true for all $\mathcal{P} \in \text{MinPlus}(X'_u)$.*

For an antidiagonal D , let D_{left} be the antidiagonal adjacent to D and to its left. Similarly let D_{right} be the antidiagonal adjacent to D and to its right.

Lemma 3.9. *Fix $\mathcal{P} \in \text{MinPlus}(X'_u)$. Fix an antidiagonal D . Let $+_a \in D$ and suppose $+_b$ is in D , D_{left} or D_{right} so that $+_b$ is weakly southwest of $+_a$. Then for any $\mathcal{P}' \in \text{MinPlus}(X'_u)$, $+_b$ is weakly southwest of $+_a$.*

Proposition 3.10. (I) *Let u be a biGrassmannian permutation and $\mathcal{P}, \mathcal{P}' \in \text{MinPlus}(X'_u)$. Consider the following lists of indices:*

SAME = $(a : +_a \text{ appears in the same location in } \mathcal{P} \text{ and } \mathcal{P}')$,

SW = $(a : +_a \text{ in } \mathcal{P}' \text{ is strictly southwest of } +_a \text{ in } \mathcal{P})$,

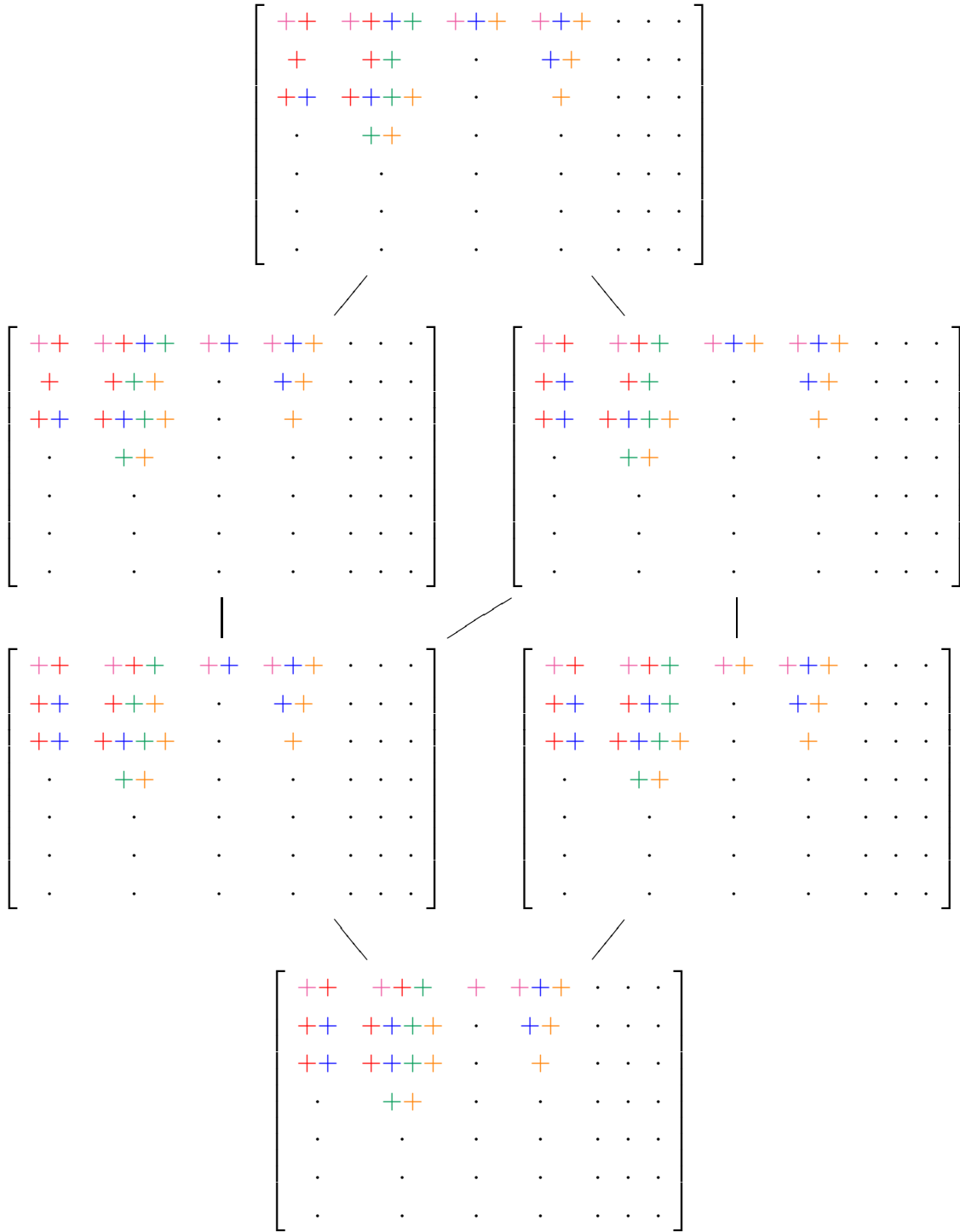


FIGURE 3. The subsubset of $\text{Multi}(5361724)$ with support \mathcal{P} (cf. Example 3.7).

and

$$\text{NE} = (a : +_a \text{ in } \mathcal{P}' \text{ is strictly northeast of } +_a \text{ in } \mathcal{P}).$$

Let Λ be the sequence contained by the concatenation SAME, SW, NE where SAME and SW are listed in increasing order whereas the elements of NE are listed in decreasing order.

Then there exists a sequence:

$$(3.6) \quad \mathcal{P} := \mathcal{P}_1 \mapsto \mathcal{P}_2 \mapsto \mathcal{P}_3 \mapsto \cdots \mapsto \mathcal{P}_h \mapsto \mathcal{P}_{h+1} \mapsto \cdots \mapsto \mathcal{P}_\ell \mapsto \mathcal{P}_{\ell+1} := \mathcal{P}'$$

where

- (i) $\ell = \ell(u) = |\lambda(u)|$
 - (ii) each $\mathcal{P}_i \in \text{MinPlus}(X'_u)$;
 - (iii) $\mathcal{P}_h \mapsto \mathcal{P}_{h+1}$ is an application of a long move to $+\Lambda_h$.
- (II) $(\text{MinPlus}(X'_u), <)$ is a lattice.

Example 3.11. Let $u = 1267345$. The ordering (3.5) is as follows (where for brevity we ignore the unused bottom three rows of the 7×7 ambient square):

$$\mathcal{D}_{\text{bot}}(u) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +_2 & +_4 & +_6 & \cdot & \cdot & \cdot & \cdot \\ +_1 & +_3 & +_5 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

$$\text{Let } \mathcal{P} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +_2 & +_4 & +_6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +_5 & \cdot & \cdot & \cdot \\ +_1 & +_3 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \text{ and } \mathcal{P}' = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & +_6 & \cdot & \cdot \\ \cdot & \cdot & +_4 & \cdot & +_5 & \cdot & \cdot \\ +_2 & \cdot & +_3 & \cdot & \cdot & \cdot & \cdot \\ +_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Here $\text{SAME} = (1, 4)$, $\text{SW} = (2)$, and $\text{NE} = (6, 5, 3)$. Therefore $\Lambda = (1, 4, 2, 6, 5, 3)$.

Removing trivial long moves for SAME , the sequence (3.6) consists of the following moves (where we have underlined $+\Lambda_h$ for emphasis):

$$\begin{aligned} \mathcal{P}_3 &= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \underline{+_2} & +_4 & +_6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +_5 & \cdot & \cdot & \cdot \\ +_1 & +_3 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \mathcal{P}_4 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +_4 & \underline{+_6} & \cdot & \cdot & \cdot \\ +_2 & \cdot & \cdot & +_5 & \cdot & \cdot & \cdot \\ +_1 & +_3 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \\ \mapsto \mathcal{P}_5 &= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & +_6 & \cdot & \cdot \\ \cdot & \cdot & +_4 & \cdot & \cdot & \cdot & \cdot \\ +_2 & \cdot & \cdot & \underline{+_5} & \cdot & \cdot & \cdot \\ +_1 & +_3 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \mathcal{P}_6 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & +_6 & \cdot & \cdot \\ \cdot & \cdot & +_4 & \cdot & +_5 & \cdot & \cdot \\ +_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +_1 & \underline{+_3} & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \\ \mapsto \mathcal{P}_7 &= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & +_6 & \cdot & \cdot \\ \cdot & \cdot & +_4 & \cdot & +_5 & \cdot & \cdot \\ +_2 & \cdot & +_3 & \cdot & \cdot & \cdot & \cdot \\ +_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{aligned}$$

In particular, $\mathcal{P}_4 = \mathcal{P}_{\#\text{SAME} + \#\text{SW} + 1}$ is $\mathcal{P} \wedge \mathcal{P}'$. □

Proof of Proposition 3.10: (I): We show that for each $1 \leq h \leq \ell$ one can give the desired long move. Clearly, we can use the trivial long move for $1 \leq h \leq \#\text{SAME}$.

For $\#\text{SAME} < h \leq \#\text{SAME} + \#\text{SW}$ we have $\Lambda_h \in \text{SW}$. Suppose the $+\Lambda_h$ in \mathcal{P}' is in row r' and suppose the $+\Lambda_h$ in \mathcal{P} is in row r (where we have assumed $r' > r$ in matrix notation). Let D be the antidiagonal that $+\Lambda_h$ sits in (in either \mathcal{P} and \mathcal{P}').

Claim 3.12. *In \mathcal{P}_h , there is no $+_b$ in rows*

- (1) $r, r+1, \dots, r'-1$ of D_{left} ;
- (2) $r+1, r+2, \dots, r'$ of D_{right} ; or
- (3) $r+1, r+2, \dots, r'$ of D .

Proof of Claim 3.12: If $\Lambda_h = b$ then by definition the unique $+_{\Lambda_h} = +_b$ is in row r of D , and in particular not in (1), (2) or (3). If $\Lambda_h < b$, by the definition of the ordering on $+$'s combined with Lemma 3.9, $+_b$ is not weakly southwest of $+_{\Lambda_h}$. Thus we may assume $\Lambda_h > b$. Since $\Lambda_h > b$ the position of $+_b$ in \mathcal{P}' will either be the same (if $b \in \text{SAME}$ or $b \in \text{SW}$), or strictly northeast of its position in \mathcal{P}_h (if $b \in \text{NE}$). The position of $+_{\Lambda_h}$ in \mathcal{P}' is row r' of D , which is weakly southwest of the position of $+_b$ in \mathcal{P}' . However, by the assumption that $+_b$ is in (1), (2) or (3), we see that in \mathcal{P}_h , $+_b$ is weakly southwest of $+_{\Lambda_h}$. Hence we obtain a contradiction of Lemma 3.9. \square

In view of Claim 3.12 we may apply the long move $\mathcal{P}_h \mapsto \mathcal{P}_{h+1}$ that moves the $+_{\Lambda_h}$ in row r of \mathcal{P}_h to row r' , showing (iii). Since each long move is by definition a composition of southwest moves, (ii) holds by Lemma 3.5(I).

Finally, for $\Lambda_h \in \text{NE}$ we have a long move for the same reasons (*mutatis mutandis*) as in our analysis above of $\Lambda_h \in \text{SW}$. (Alternatively, let $\mathcal{W}' = \mathcal{P}_{\#\text{SAME} + \#\text{SW} + 1}$ and $\mathcal{W} = \mathcal{P}'$. Then by the above arguments there are southwest long moves connecting \mathcal{W} to \mathcal{W}' . Then we can reverse these moves to give the desired northeast long moves from \mathcal{W}' to \mathcal{P}' .)

Since the list SAME, SW, NE is of length ℓ , (i) holds trivially.

(II): We will only construct $\mathcal{P} \wedge \mathcal{P}' \in \text{MinPlus}(X'_u)$ (the construction of $\mathcal{P} \vee \mathcal{P}'$ is entirely analogous). From (I) we have the sequence of long moves (3.6) that transform \mathcal{P} into \mathcal{P}' . Let $\mathcal{R} = \mathcal{P}_h$, where h corresponds to the first index in NE, i.e., $h = \#\text{SAME} + \#\text{SW} + 1$.

\mathcal{R} is obtained from \mathcal{P} by applying a series of southwest long moves and $\mathcal{R} < \mathcal{P}$. Likewise, \mathcal{P}' is obtained from \mathcal{R} entirely by northeast long moves, so $\mathcal{R} < \mathcal{P}'$. Suppose we have some other $<$ -lower bound \mathcal{S} of \mathcal{P} and \mathcal{P}' . We may construct \mathcal{T} , a $<$ -lower bound for \mathcal{R} and \mathcal{S} , by the same argument above we have used to construct \mathcal{R} from \mathcal{P} and \mathcal{P}' .

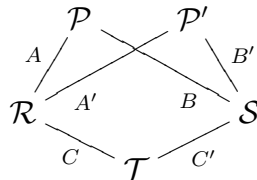
That $\mathcal{R} = \mathcal{P} \wedge \mathcal{P}'$ is immediate from the following:

Claim 3.13. $\mathcal{R} \geq \mathcal{T} = \mathcal{S}$.

Proof of Claim 3.13: We will work with the sets:

$$\begin{aligned} A &= \{a : +_a \text{ in } \mathcal{R} \text{ is strictly southwest of } +_a \text{ in } \mathcal{P}\} \\ A' &= \{a : +_a \text{ in } \mathcal{R} \text{ is strictly southwest of } +_a \text{ in } \mathcal{P}'\} \\ B &= \{a : +_a \text{ in } \mathcal{S} \text{ is strictly southwest of } +_a \text{ in } \mathcal{P}\} \\ B' &= \{a : +_a \text{ in } \mathcal{S} \text{ is strictly southwest of } +_a \text{ in } \mathcal{P}'\} \\ C &= \{a : +_a \text{ in } \mathcal{T} \text{ is strictly southwest of } +_a \text{ in } \mathcal{R}\} \\ C' &= \{a : +_a \text{ in } \mathcal{T} \text{ is strictly southwest of } +_a \text{ in } \mathcal{S}\} \end{aligned}$$

Summarizing, we have:



Since $\mathcal{T} < \mathcal{R} < \mathcal{P}$ and $\mathcal{T} < \mathcal{S} < \mathcal{P}$,

$$(3.7) \quad A \cup C = \{a : +_a \text{ in } \mathcal{T} \text{ is strictly southwest of } +_a \text{ in } \mathcal{P}\} = B \cup C'.$$

Similarly,

$$(3.8) \quad A' \cup C = B' \cup C'.$$

By the construction of \mathcal{R} and \mathcal{T} we have

$$(3.9) \quad C \cap C' = \emptyset$$

and

$$(3.10) \quad A \cap A' = \emptyset.$$

Intersecting both sides of (3.7) by C' gives:

$$(A \cup C) \cap C' = (B \cup C') \cap C' \iff (A \cap C') \cup (C \cap C') = (B \cap C') \cup (C' \cap C').$$

By (3.9) we have $A \cap C' = C'$, which in turn implies $C' \subseteq A$. Likewise, (3.8) implies $C' \subseteq A'$. Therefore by (3.10), $C' = \emptyset$ holds. This shows $\mathcal{R} \geq \mathcal{T} = \mathcal{S}$, as claimed. \square

This completes the proof of the proposition. \square

Lemma 3.14. Fix $\mathcal{P}, \mathcal{P}' \in \text{MinPlus}(X'_u)$ (where u is biGrassmannian), and let

$$\mathcal{P} =: \mathcal{P}_1 \mapsto \mathcal{P}_2 \mapsto \dots \mapsto \mathcal{P}_h \mapsto \mathcal{P}_{h+1} \mapsto \dots \mapsto \mathcal{P}_\ell \mapsto \mathcal{P}_{\ell+1} := \mathcal{P}'$$

be the sequence (3.6) in Proposition 3.10 (II). Then for any $1 \leq h \leq \ell + 1$, $\mathcal{P}_h \subseteq \mathcal{P} \cup \mathcal{P}'$. In particular, if $\mathcal{R} = \mathcal{P} \wedge \mathcal{P}'$ in the lattice $(\text{MinPlus}(X'_u), <)$ (where u is biGrassmannian), then $\mathcal{R} \subseteq \mathcal{P} \cup \mathcal{P}'$.

Proof. The claim about \mathcal{P}_h follows from the construction of the sequence (3.6) in Proposition 3.10(II). The claim about \mathcal{R} holds since in the proof of Proposition 3.10(II), we have shown $\mathcal{R} = \mathcal{P}_h$ (where $h = \#\text{SAME} + \#\text{SW} + 1$). \square

3.4. Proof of Theorem 3.6. (I): Fix $w \in S_n$, $\mathcal{P} \in \text{MinPlus}(X'_w)$. Let

$$\mathcal{Q} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k), \mathcal{Q}' = (\mathcal{P}'_1, \mathcal{P}'_2, \dots, \mathcal{P}'_k) \in \text{Multi}(\mathcal{P}).$$

By Proposition 3.10, we may connect \mathcal{P}_1 to \mathcal{P}'_1 by long moves, as in (3.6):

$$\mathcal{P}_1 = \mathcal{P}_{1,1} \mapsto \mathcal{P}_{1,2} \mapsto \dots \mapsto \mathcal{P}_{1,h} \mapsto \mathcal{P}_{1,h+1} \mapsto \dots \mapsto \mathcal{P}_{1,\ell+1} = \mathcal{P}'_1.$$

By Lemma 3.14, $\mathcal{P}_{1,h} \subseteq \mathcal{P}_1 \cup \mathcal{P}'_1 \subseteq \mathcal{P}$. Hence

$$\text{supp}(\mathcal{P}_{1,h}, \mathcal{P}_2, \dots, \mathcal{P}_k) \subseteq \mathcal{P}.$$

Since $\mathcal{P} \in \text{MinPlus}(X'_w)$, this containment is an equality. That is, each $(\mathcal{P}_{1,h}, \mathcal{P}_2, \dots, \mathcal{P}_k) \in \text{Multi}(\mathcal{P})$. In the case $h = \ell + 1$, we reach $(\mathcal{P}'_1, \mathcal{P}_2, \mathcal{P}_3, \dots, \mathcal{P}_k)$ from $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \dots, \mathcal{P}_k)$.

Continuing in this manner, one connects

$$(\mathcal{P}'_1, \mathcal{P}_2, \mathcal{P}_3, \dots, \mathcal{P}_k) \text{ to } (\mathcal{P}'_1, \mathcal{P}'_2, \mathcal{P}_3, \dots, \mathcal{P}_k),$$

by long moves, that keep one in $\text{Multi}(\mathcal{P})$, until one reaches $(\mathcal{P}'_1, \mathcal{P}'_2, \dots, \mathcal{P}'_k)$.

(II): This is Proposition 3.10(II).

(III): Let $\mathcal{Q}, \mathcal{Q}' \in \text{Multi}(\mathcal{P})$, as in (I). Since by definition $\text{Multi}(\mathcal{P})$ is a subposet of the lattice $\text{Multi}(w)$, it suffices to show $\mathcal{Q} \wedge \mathcal{Q}' \in \text{Multi}(\mathcal{P})$. (The argument for the join is

similar.) By Proposition 3.10, for each i , there is $\mathcal{R}_i = \mathcal{P}_i \wedge \mathcal{P}'_i$. In general, the meet in a Cartesian product of lattices is formed by taking the meet in each component. Therefore,

$$\mathcal{Q} \wedge \mathcal{Q}' = (\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k) \in \text{Multi}(w).$$

By Lemma 3.14, $\mathcal{R}_i \subseteq \mathcal{P}_i \cup \mathcal{P}'_i \subseteq \mathcal{P}$. Hence $\text{supp}(\mathcal{Q} \wedge \mathcal{Q}') \subseteq \mathcal{P}$. However, since $\mathcal{P} \in \text{MinPlus}(X'_w)$, we must have $\text{supp}(\mathcal{Q} \wedge \mathcal{Q}') = \mathcal{P}$, i.e., $\mathcal{Q} \wedge \mathcal{Q}' \in \text{Multi}(\mathcal{P})$, as desired. \square

The “MinPlus” hypothesis of Theorem 3.6 is necessary, as we now demonstrate:

Example 3.15 ($\text{Multi}(\mathcal{P})$ for non minimal plus diagrams). Let $w = 14253$. Then we have $\text{biGrass}(w) = \{14235, 12453\}$. Let

$$\mathcal{P} = \begin{bmatrix} \cdot & \cdot & + & \cdot & \cdot \\ + & + & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in \text{Plus}(X'_w) \setminus \text{MinPlus}(X'_w).$$

Observe

$$\text{Multi}(\mathcal{P}) = \left\{ \begin{bmatrix} \cdot & \cdot & \textcolor{violet}{+} & \cdot & \cdot \\ \textcolor{blue}{+} & \textcolor{blue}{+} & \cdot & \cdot & \cdot \\ \cdot & \textcolor{violet}{+} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & \textcolor{blue}{+} & \cdot & \cdot \\ \textcolor{blue}{+} & \textcolor{violet}{+} & \cdot & \cdot & \cdot \\ \cdot & \textcolor{violet}{+} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

$\text{Multi}(\mathcal{P})$ consists of two incomparable elements, so is in particular not a lattice. \square

3.5. Conclusion of the proof of Theorem 1.1. By Lemma 2.2(II), $\text{Multi}(\mathcal{P}) \neq \emptyset$. In addition, by Theorem 3.6, $\text{Multi}(\mathcal{P})$ is a finite lattice and thus has a unique minimum $\mathcal{M}_{\mathcal{P}}$. Let

$$\overline{\text{Multi}(w)} := \{\mathcal{M}_{\mathcal{P}} : \mathcal{P} \in \text{MinPlus}(X'_w)\}.$$

Hence, trivially, we have a bijection

$$\Psi : \overline{\text{Multi}(w)} \rightarrow \text{MinPlus}(X'_w).$$

Let $\text{AllPrism}(w)$ denote the set of *all* prism tableaux and $\text{MinPrism}(w)$ the set of minimal prism tableaux for w . From the definitions,

$$\text{Prism}(w) \subseteq \text{MinPrism}(w) \subseteq \text{AllPrism}(w).$$

Claim 3.16. *There is a bijection $\Phi : \text{AllPrism}(w) \rightarrow \text{Multi}(w)$.*

Proof. We associate each $\mathcal{P} \in \text{MinPlus}(X'_{u_e})$ with a filling of R_e . To do this, notice that by definition R_e sits in $n \times n$ exactly as the $+$'s of $\mathcal{D}_{\text{bot}}(u_e)$ do. Hence by Lemma 3.5(II) there is a bijection between the $+$'s of \mathcal{P} and the boxes of R_e .

Assign to each box of R_e the colored label that is the row position of that boxes' associated $+$ in \mathcal{P} . Then these labels of color e satisfy (S1) by definition. That they satisfy (S2) and (S3) follows from Lemma 3.8. Finally, (S4) holds by Lemma 3.5(II).

The map we have just described from $\mathcal{P} \in \text{MinPlus}(X'_{u_e})$ and the (S1)-(S4) fillings \mathcal{S} of R_e is clearly injective. That it is a surjection follows since the tableaux \mathcal{S} are clearly

in weight-preserving bijection with the semistandard tableaux for \mathfrak{S}_{u_e} [Ma01, Proposition 2.6.8], which are known to be in bijection with $\mathcal{P} \in \text{MinPlus}(X'_{u_e})$, see, e.g., [KnMiYo09, Proposition 5.3] (and for an earlier reference, see [Ko00]).

Now, given $T \in \text{AllPrism}(w)$ we construct $\Phi(T) := (\mathcal{P}_1, \dots, \mathcal{P}_k) \in \text{Multi}(w)$ by applying the above correspondence independently to each R_e . That this is a bijective map follows from the bijectivity on each component. \square

Corollary 3.17. Φ restricts to a bijection $\tilde{\Phi} : \text{MinPrism}(w) \rightarrow \text{supp}^{-1}(\text{MinPlus}(X'_w)) \subseteq \text{Multi}(w)$.

Proof. Since Φ is a bijection, we are only required to show that

$$\text{im } \Phi|_{\text{MinPrism}(w)} = \text{supp}^{-1}(\text{MinPlus}(X'_w)).$$

However, this holds, since a tableau $T \in \text{AllPrism}(w)$ is in $\text{MinPrism}(w)$ if and only if $\text{supp}(\Phi(T))$ has cardinality $\ell(w)$, i.e. if and only if $\text{supp}(\Phi(T)) \in \text{MinPlus}(X'_w)$. \square

Claim 3.18. Let $\mathcal{Q} \in \text{Multi}(w)$ and let $T = \Phi^{-1}(\mathcal{Q})$. Then T has an unstable triple if and only if there exists a southwest long move $\mathcal{Q} \mapsto \mathcal{Q}'$ such that $\text{supp}(\mathcal{Q}) = \text{supp}(\mathcal{Q}')$.

Proof of Claim 3.18: (\Rightarrow) Suppose T has an unstable triple $\{\ell_c, \ell_d, \ell'_e\}$ contained in an antidiagonal D . Let T' be the tableau obtained by replacing ℓ_c with ℓ'_c . We must show \mathcal{Q} differs from $\mathcal{Q}' := \Phi(T')$ by a southwest long move such that $\text{supp}(\mathcal{Q}) = \text{supp}(\mathcal{Q}')$. If one could not conduct a southwest long move, there must have been some $+$ of color c in the region consisting of:

- (1) rows $\ell, \ell + 1, \dots, \ell' - 1$ of D_{left} .
- (2) rows $\ell + 1, \ell + 2, \dots, \ell'$ of D_{right} .
- (3) rows $\ell + 1, \ell + 2, \dots, \ell'$ of D .

Moving the $+$ of color c in row ℓ to row ℓ' would cause it to appear southwest of $+$, contradicting Lemma 3.9.

So now assume we have a southwest long move. It remains to check the support assertion. The labels ℓ_c and ℓ_d in T each ensure there is a $+$ in row ℓ of D in $\text{supp}(\mathcal{Q})$, while ℓ'_e gives a $+$ to row ℓ' of D in $\text{supp}(\mathcal{Q})$. Similarly, ℓ_d in T' corresponds to a plus in row ℓ of D in $\text{supp}(\mathcal{Q}')$, while ℓ'_c and ℓ'_e gives each ensure there is a $+$ in row ℓ' of D in $\text{supp}(\mathcal{Q}')$. So replacing ℓ_c in T with ℓ'_c in T' gives $\text{supp}(\mathcal{Q}) = \text{supp}(\mathcal{Q}')$.

(\Leftarrow) Suppose we may apply a support preserving southwest long move to

$$\mathcal{Q} = (\mathcal{P}_1, \dots, \mathcal{P}_c, \dots, \mathcal{P}_k) \mapsto \mathcal{Q}' = (\mathcal{P}_1, \dots, \mathcal{P}'_c, \dots, \mathcal{P}_k).$$

That is, there is an antidiagonal $D \subset n \times n$ such that \mathcal{P}_c contains a $+$ in row ℓ of D that may be moved to row $\ell' > \ell$ by a southwest long move. Since $\text{supp}(\mathcal{Q}) = \text{supp}(\mathcal{Q}')$, there must be colors d, e with the property that \mathcal{P}_d has a $+$ in row ℓ of D and \mathcal{P}_e has a $+$ in row $\ell' > \ell$ of D . In T , this implies that there are labels $\{\ell_c, \ell_d, \ell'_e\}$ in D . Let T' be obtained from T by replacing ℓ_c with ℓ'_c . Then $T' = \Phi(\mathcal{Q}) \in \text{AllPrism}(w)$. So $\{\ell_c, \ell_d, \ell'_e\}$ is an unstable triple. \square

Claim 3.19. Φ (further) restricts to a bijection, $\hat{\Phi} : \text{Prism}(w) \rightarrow \overline{\text{Multi}(w)}$.

Proof. Since we know Φ is a bijection, we need only show that $\text{im } \Phi|_{\text{Prism}(w)} = \overline{\text{Multi}(w)}$.

If $\mathcal{M}_{\mathcal{P}} \in \overline{\text{Multi}(w)}$, then $\mathcal{M}_{\mathcal{P}}$ is by definition the minimum in $\text{Multi}(\mathcal{P})$. Let $T := \tilde{\Phi}^{-1}(\mathcal{M}_{\mathcal{P}})$ (this exists by Corollary 3.17). By Claim 3.18 (\Rightarrow), if T has an unstable triple,

then there exists a southwest long move $\mathcal{M}_{\mathcal{P}} \mapsto \mathcal{Q}'$ so that $\text{supp}(\mathcal{M}_{\mathcal{P}}) = \text{supp}(\mathcal{Q}')$. But then $\mathcal{Q}' < \mathcal{M}_{\mathcal{P}}$, a contradiction. Hence $T \in \text{Prism}(w)$. Thus, $\text{im } \Phi|_{\text{Prism}(w)} \supseteq \overline{\text{Multi}(w)}$.

Suppose $T \in \text{Prism}(w) \subseteq \text{MinPrism}(w)$, and let $\mathcal{Q} := \tilde{\Phi}(T) \in \text{supp}^{-1}(\text{MinPlus}(X'_w))$. Suppose \mathcal{Q} is not the minimum element in $\text{Multi}(\text{supp}(\mathcal{Q}))$. Then there exists a southwest long move $\mathcal{Q} \mapsto \mathcal{Q}'$, so that $\text{supp}(\mathcal{Q}) = \text{supp}(\mathcal{Q}')$. Then by Claim 3.18 (\Leftarrow), T must have had an unstable triple, contradicting $T \in \text{Prism}(w)$. Thus, $\mathcal{Q} = \mathcal{M}_{\text{supp}(\mathcal{Q})} \in \overline{\text{Multi}(w)}$. This shows $\text{im } \Phi|_{\text{Prism}(w)} \subseteq \overline{\text{Multi}(w)}$, as required. \square

By Claim 3.19, $\Psi \circ \widehat{\Phi} : \text{Prism}(w) \rightarrow \text{MinPlus}(X'_w)$ is a bijection. Now,

$$\text{wt}(T) = \prod_i x_i^{\# \text{ of antidiagonals containing } i} \quad \text{and} \quad \text{wt}(\mathcal{P}) = \prod_i x_i^{\# \text{ of } +\text{'s in row } i}.$$

That $\text{wt}(T) = \text{wt}((\Psi \circ \widehat{\Phi})(T))$ is immediate from these definitions. Hence the theorem follows. \square

4. FURTHER DISCUSSION

4.1. Comparisons to the literature. Ultimately, the evaluation of any model for Schubert polynomial rests on its success towards the *Schubert problem*, i.e., finding a generalized Littlewood-Richardson rule for Schubert polynomials. Due to the analogy with Sym, one hopes that a solution will not only provide *merely* a rule, but rather lead to an entire companion combinatorial theory. This would presumably enrich our understanding of Pol and its role in mathematics just as the Young tableau theory does for Sym.

That the prism model manifestly uses Young tableaux is our impetus for ongoing investigations that fundamental tableaux algorithms might admit prism-generalizations.

The first rule for Schubert polynomials was conjectured by [Ko90]. This rule begins with the diagram of w and evolves other subsets of $n \times n$ by a simple move, the Schubert polynomial is a generating series over these subsets. A proof is presented in [Wi99, Wi02]. Arguably, this rule is the most handy of all known rules, even though the set of Kohnert diagrams does not have a closed description.

Probably the most well-known and utilized formula is given by [BiJoSt93], which expresses the Schubert polynomial in terms of reduced decompositions of w . This rule is made graphical by the *RC*-graphs of [BeBi93] (cf. [FoKi96]). One can obtain any *RC*-graph for w from any other by the *chute* and *ladder* moves of [BeBi93].

While neither of the above rules transparently reduces to the tableau rule for Schur polynomials, it is not too difficult to show in either case, that the objects involved do biject with semistandard tableaux, see [Ko90] and [Ko00] respectively.

We are not aware of any published bijection between the Kohnert rule and any other model for Schubert polynomials. On the other hand, there is a map between the prism tableaux and *RC*-graphs: the labels on the i -th antidiagonal indicate the row position of the $+$'s on the same antidiagonal in the associated *RC*-graph. This map is clearly injective but we do not currently have a purely *combinatorial* proof that the map is well-defined. Tracing our proof of the main theorem, well-definedness comes from the Gröbner basis theorem of [KnMi05]. Moreover, in said proof, we treat each *RC*-graph as a *specific* overlay of *RC*-graphs for bigrassmannian permutations. The latter *RC*-graphs are in bijection

with semistandard tableaux of rectangular shape. This is the reason for the “dispersion” remark of the introduction.

The work of [FoGrReSh97] gives a tableau rule for Schubert polynomials of a different flavor. This rule treats \mathfrak{S}_w as a generating series for **balanced fillings** of the diagram of w . The reduction to semistandard tableaux for Grassmannian w seems non-trivial.

In [BuKrTaYo04], a formula is given for a Schubert polynomial as a nonnegative integer linear combination of sum of products of Schur functions in disjoint sets of variables (with nontrivial coefficients). This is also in some sense a tableau formula for \mathfrak{S}_w . In [Le04] this result is rederived as a consequence of the crystal graph structure on RC -graphs developed there.

4.2. Details of the reduction to semistandard tableaux. We now explicate the reduction from prism tableaux to ordinary semistandard tableaux, as indicated in the introduction.

Proposition 4.1. *Assume $w \in S_n$ is Grassmannian.*

- (I) *The shape $\lambda(w)$ is a Young diagram, in French notation.*
- (II) *Let $T \in \text{MinPrism}(w)$. All labels of a box of T have the same number.*
- (III) *T does not have unstable triples, i.e., $\text{MinPrism}(w) = \text{Prism}(w)$.*

Proof. (I): Since w is Grassmannian, it has a unique descent, $w(k) > w(k+1)$. Furthermore, all essential boxes of w lie in the k th row, say in columns $a_1 < \dots < a_j$. The rectangle $R_{(k,a_i)}$ starts at row $r_w(k, a_i) + 1$, which strictly increases as i increases, since essential boxes to the right in the diagram take on higher values for the rank function. Each rectangle is left justified by construction, and has width $a_i - r_w(k, a_i)$. This value strictly increases with each i , since

$$a_i + (r_w(k, a_{i+1}) - r_w(k, a_i)) < a_{i+1},$$

as seen from the diagram of w . So the rectangles $R_{(k,a_i)}$ overlap to form the shape of a partition.

(II): Suppose not. Let x be a “bad” box, i.e., one with ℓ_c and ℓ'_d in x where $\ell \neq \ell'$ (and thus $c \neq d$). We may assume x is the northeast-most bad box. Let D be the antidiagonal containing x . We may also assume that each box of D contains a label of color c .

Case 1: ($\ell' > \ell$): Then by (S2) and (S3) the labels of color c in D , that are strictly northeast of x , are all distinct and different than both ℓ and ℓ' . For the same reason, all labels of color d in D strictly southwest of x are distinct and different than ℓ and ℓ' . Hence the total number of distinct numbers in D exceeds $\#D$. Since $|\lambda(w)| = \ell(w)$, we conclude T cannot be minimal, a contradiction.

Case 2: ($\ell' < \ell$): Again by (S2) and (S3), all labels of color c that are in D but strictly southwest of x are distinct and are also different than ℓ and ℓ' . Hence if D is not overfull (i.e., has more distinct numbers than boxes) ℓ'_c must appear in a box y in D that is strictly northeast of x . By the definition of prism tableaux and $\lambda(w)$, either there exists:

- (1) a box z in the row of x and in the column of y that contains labels m_c and m'_d , or
- (2) a box w in the column of x and in the row of y that contains labels m_c and m'_d .

We may assume the first case occurs (the argument for the other case is the same). In view of the $\ell'_c \in y$ combined with (S3), $m > \ell'$. On the other hand, in view of the $\ell'_d \in x$

combined with (S2), $m' \leq \ell'$. Hence we see z is a bad box strictly east of x , a contradiction of the extremality of x .

(III): Suppose T has an unstable triple $\{\ell_c, \ell_d, \ell'_e\}$ in antidiagonal D . Let T' be the tableaux obtained by replacing ℓ_c with ℓ'_e . Then by definition, $T' \in \text{Prism}(w)$. By (II), ℓ_c must sit in a box containing no other labels. By the definition of $\lambda(w)$, this furthermore implies every box of D in $\lambda(w)$ has a label of color c . (II) then implies the box containing ℓ'_e must contain a label ℓ'_c . This contradicts (S2) and (S3) combined. \square

4.3. Stable Schubert polynomials. The **stable Schubert polynomial** (also known as the *Stanley symmetric polynomial*) is the generating series defined by

$$F_w(x_1, x_2, \dots) := \lim_{m \rightarrow \infty} \mathfrak{S}_{1^m \times w},$$

where if $w \in S_n$ then $1^m \times w$ is the permutation in S_{m+n} defined by

$$(1^m \times w)(i) = i \text{ for } 1 \leq i \leq m \text{ and } (1^m \times w)(m+i) = m+w(i) \text{ for } 1 \leq i \leq n.$$

It is true that

$$F_w(x_1, x_2, \dots, x_m, 0, 0, \dots) = \mathfrak{S}_{1^m \times w}(x_1, \dots, x_m, 0, 0, \dots).$$

Now, notice that $\lambda(1^m \times w)$ and $\lambda(w)$ are the same shape, but the former is shifted down m steps in the grid relative to $\lambda(w)$. Therefore it follows that

$$F_w(x_1, x_2, \dots, x_m, 0, 0, \dots) = \sum_T \text{wt}(T),$$

where the sum is over all *unflagged* (i.e., exclude (S4)) minimal prism tableaux of shape $\lambda(w)$ that use the labels $1, 2, \dots, m$. In the limit, this argument implies the generating series $F_w(x_1, x_2, \dots)$ is given by the same formula, except we allow all labels from \mathbb{N} .

4.4. An overlay interpretation of chute and ladder moves. In [BeBi93], **chute moves** were defined for pipe dreams. These moves are locally of the form

$$(4.1) \quad \mathcal{P} = \begin{array}{cccccccc} \cdot & + & + & + & \cdots & + & + & \cdot \\ + & + & + & + & \cdots & + & + & \cdot \end{array} \rightarrow \begin{array}{cccccccc} \cdot & + & + & + & \cdots & + & + & + \\ \cdot & + & + & + & \cdots & + & + & \cdot \end{array} = \mathcal{Q}$$

Suppose $\mathcal{P} \in \text{MinPlus}(X'_w)$, $\text{biGrass}(w) = \{u_1, \dots, u_k\}$ and $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$, where $\mathcal{P}_i \in \text{MinPlus}(X'_{u_i})$. We now show:

The chute move's "long jump" of a single $+$ may be interpreted as a sequence of the northeast local moves (3.3) applied to the \mathcal{P}_i 's.

Example 4.2. Let $w = 1432$. Now, $\text{biGrass}(w) = \{u_1 = 1423, u_2 = 1342\}$. Consider the following sequence of northeast moves

$$\begin{bmatrix} \cdot & + & \cdot & \cdot \\ + & ++ & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & + & + & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & ++ & + & \cdot \\ \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Let the support of the first and third plus diagrams be \mathcal{P} and \mathcal{Q} , respectively. We have $\mathcal{P}, \mathcal{Q} \in \text{MinPlus}(X'_w)$. \mathcal{P} and \mathcal{Q} differ by a chute move. At the level of the overlays, one sees this transition as an application of (3.3) to each blue $+$ in the second row. \square

Example 4.2 indicates the general pattern. Let (i, j) be the position of the southwest $+$ of \mathcal{P} in (4.1) and $(i - 1, j')$ the position of the northeast $+$ of \mathcal{Q} in (4.1). Without loss of generality, we may assume each $\mathcal{P}_1, \dots, \mathcal{P}_t$ contains a $+$ at (i, j) while all other \mathcal{P}_h do not.

Claim 4.3. *Consider the interval of consecutive $+$'s in row i of \mathcal{P}_h ($1 \leq h \leq t$) starting at the left with the $+$ in position (i, j) . One can apply the move (3.4) (in the right to left order) to each of these $+$'s to obtain $\mathcal{P}'_1, \dots, \mathcal{P}'_t$.*

Proof. It follows from Lemma 3.9 that the configurations given below do not appear in $\text{MinPlus}(X'_u)$ whenever u is biGrassmannian:

$$\left\{ \begin{bmatrix} \cdot & + \\ + & + \end{bmatrix}, \begin{bmatrix} \cdot & + \\ + & \cdot \end{bmatrix}, \begin{bmatrix} + & + \\ + & \cdot \end{bmatrix} \right\}.$$

Suppose there is an obstruction to one of the local moves. It must appear in row $i - 1$. Due to the $+$ in position (i, j) , such an obstruction necessarily forces one of the above configurations to appear, causing a contradiction. \square

Claim 4.4. $\mathcal{P}'_h \subseteq \mathcal{Q}$ for $1 \leq h \leq t$ and $\mathcal{P}_h \subseteq \mathcal{Q}$ for $t + 1 \leq h \leq k$.

Proof. First suppose $1 \leq h \leq t$. Each move from Claim 4.3 takes a $+$ from position (i, a) with $j \leq a < j'$ and replaces it with a $+$ in position $(i - 1, a + 1) \in \mathcal{Q}$. Furthermore, each \mathcal{P}'_h has no $+$ in position (i, j) . So $\mathcal{P}'_h \subseteq \mathcal{Q}$. If $h \geq t + 1$, then by assumption \mathcal{P}_h has no $+$ in position (i, j) . So $\mathcal{P}_h \subseteq \mathcal{P} \setminus \{(i, j)\} \subseteq \mathcal{Q}$. \square

Claim 4.5. $\mathcal{Q} = \mathcal{P}'_1 \cup \dots \cup \mathcal{P}'_t \cup \mathcal{P}_{t+1} \cup \dots \cup \mathcal{P}_k$.

Proof. Let $\tilde{\mathcal{Q}} = \mathcal{P}'_1 \cup \dots \cup \mathcal{P}'_t \cup \mathcal{P}_{t+1} \cup \dots \cup \mathcal{P}_k$. Suppose $\tilde{\mathcal{Q}} \neq \mathcal{Q}$. By Claim 4.4, each $\mathcal{P}'_i \subseteq \mathcal{Q}$, for $1 \leq i \leq t$ and $\mathcal{P}_i \subseteq \mathcal{Q}$ for $t + 1 \leq i \leq k$. Then $\mathcal{Q} \supsetneq \tilde{\mathcal{Q}} \in \text{Plus}(X'_w)$, contradicting the assumption that $\mathcal{Q} \in \text{MinPlus}(X'_w)$. \square

A similar discussion applies to the ladder moves.

4.5. Future work. It is straightforward to assign weights to prism tableau in order to give a formula for double Schubert polynomials.

A generalization to Grothendieck polynomials requires a deeper control of the overlay procedure. In investigating this, one is led to some results of possibly independent interest.

Specifically, for Theorem 1.1, we have used the fact that the facets of $\Delta_{X'_w}$ are intersections of facets of those associated to $\text{biGrass}(w)$. One can make a similar conjecture for all interior faces w 's complex. Each $\Delta_{X'_w}$ is a ball or sphere [KnMi04, Theorem 3.7]. Hence one can refer to the interior faces of this complex. Let

$$\text{IntPlus}(w) = \{\mathcal{P} : \mathcal{P} \in \text{Plus}(w) \text{ and } \mathcal{F}_{\mathcal{P}} \text{ is an interior face of } \Delta_{X'_w}\}.$$

Conjecture 4.6. $\text{IntPlus}(w) \subseteq \{\mathcal{P}_1 \cup \dots \cup \mathcal{P}_k : \mathcal{P}_i \in \text{IntPlus}(u_i), \text{ for } u_i \in \text{biGrass}(w)\}.$

Conjecture 4.6 has been exhaustively computer checked for all $n \leq 6$.

As part of an intended proof of Conjecture 4.6, one defines K -theoretic analogues of the chute and ladder moves of [BeBi93]: that is if $\mathcal{P} \rightarrow \mathcal{Q}$ by a chute move (respectively, ladder move) then $\mathcal{P} \rightarrow \mathcal{P} \cup \mathcal{Q}$ is a K -chute (respectively, K -ladder move). Whereas not

all interior plus diagrams are connected by the original chute and ladder moves, it is true that they are connected once one allows the extended moves.

The first author plans to address these and related issues elsewhere.

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REFERENCES

- [BeBi93] N. Bergeron and S. Billey, *RC-graphs and Schubert polynomials*, Experimental Mathematics. Volume 2, Issue 4 (1993), 257–269.
- [BeSo02] N. Bergeron and F. Sottile, *Skew Schubert functions and the Pieri formula for flag manifolds*, Trans. Amer. Math. Soc. **354** (2002), 651–673.
- [BeSo98] ———, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J. **95** (1998), no. 2, 373–423.
- [BiJoSt93] S. Billey, W. Jockush and R. Stanley, *Some combinatorial properties of Schubert polynomials*, J. of Algebraic Comb., Vol. 2, Num. 4, 1993, 345–374.
- [BuKrTaYo04] A. Buch, A. Kresch, H. Tamvakis and A. Yong, *Schubert Polynomials and Quiver Formulas*, Duke Math. J., Vol. 122 (2004), no. 1, 125–143.
- [CoTa13] O. Coşkun and M. Taşkin, *Tower tableaux and Schubert polynomials*, J. Combin. Theory Series A, Volume 120 Issue 8 (2013), 1976–1995.
- [FoGrReSh97] S. Fomin, C. Greene, V. Reiner and M. Shimozono, *Balanced labellings and Schubert polynomials*, European Journal of Combinatorics **18** (1997), 373–389.
- [FoKi96] S. Fomin and A. N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Mathematics **153** (1996), 123–143.
- [FoSt94] S. Fomin and R. P. Stanley, *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. **103** (1994), 196–207.
- [Fu91] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J., Vol. 65 (1991), No. 3., 381–420.
- [KaSt95] M. Kalkbrener and B. Sturmfels, *Initial complexes of prime ideals*, Adv. Math. **116** (1995), 365–376.
- [Kn09] A. Knutson, *Frobenius splitting, point counting, and degeneration*, preprint, 2009. arXiv:0911.4941
- [KnMi05] A. Knutson and E. Miller, *Gröbner geometry of Schubert polynomials*, Annals. Math. **161** (2005), 1245–1318.
- [KnMi04] ———, *Subword complexes in Coxeter groups*, Adv. Math. **184** (2004), 161–176.
- [KnMiYo09] A. Knutson, E. Miller and A. Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, J. Reine Angew. Math. **630** (2009), 1–31.
- [Ko00] M. Kogan, *Schubert geometry of flag varieties and Gelfand-Cetlin theory*, Ph.D. thesis, Massachusetts Institute of Technology, 2000.
- [Ko90] A. Kohnert, *Weintrauben, Polynome, Tableaux*, Bayreuth Math. Schrift. **38** (1990), 1–97.
- [LaSh96] A. Lascoux and M.-P. Schützenberger, *Trellis et bases des groupes de Coxeter*, Electron. J. Combin. **3** (1996), no. 2, R27, 35 pp. (electronic)
- [LaSh85] ———, *Schubert polynomials and the Littlewood-Richardson rule*, Lett. Math. Phys. **10** (1985), no. 2-3, 111–124.
- [LaSh82a] ———, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 629–633.
- [LaSc82b] ———, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris, **295** (1982), 629–633.
- [Le04] C. Lenart, *A unified approach to combinatorial formulas for Schubert polynomials*, J. Algebraic Combin., **20** (2004), 263–299.

- [Ma98] P. Magyar, *Schubert polynomials and Bott-Samelson varieties*, Commentarii Mathematici Helvetici **73** (1998), 603–636.
- [Ma01] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, American Mathematical Society, Providence, RI, 2001.
- [MiSt05] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Springer Science+Business Media, Inc., 2005.
- [Wi02] R. Winkel, *A derivation of Kohnert’s algorithm from Monk’s rule*, Sémin. Lothar. Combin. **48** (2002), Art. B48f, 14 pp.
- [Wi99] ———, *Diagram rules for the generation of Schubert polynomials*, J. Combin. Th. A., **86** (1999), 14–48.

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